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THE FREE LOCAL SEMILATTICE ON A SET

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1. Introduction

Following McAlister [3] we call a semigroup S *locally inverse* if S is regular and eSe is an inverse subsemigroup of S for each idempotent e of S. Such semigroups were introduced and studied by Nambooripad [10] who called them 'pseudo-inverse' semigroups. Relative to the 'basic products', the set of idempotents of a regular semigroup forms a partial binary algebra which has been axiomatically characterized as a 'biordered set' by Nambooripad [9]. The biordered set of idempotents of a locally inverse semigroup is called a *local semilattice* in this paper. Local semilattices were called 'partially associative pseudo-semilattices' by Nambooripad [10] and 'pseudo-semilattices' by some subsequent authors [1], [5], [6], [7]. The present author used the terminology 'local semilattice' in [4].

A local semilattice may be viewed in several equivalent ways:

(i) as a biordered set E in which $|S(e,f)|=1$ for all $e,f\in E$ (see Nambooripad [9], [10]);

(ii) as a biordered set E in which $\omega(e)$ is a semilattice for each $e\in E$;

(iii) as a set E, together with two quasi-orders ω^r and ω^l which satisfy conditions (PA1) and (PA2) of [10] and their duals;

(iv) as a binary algebra $(E,\wedge)$ in which $\wedge$ satisfies the following identities and their duals:

(a) $x\wedge x=x$,

(b) $(x\wedge y)\wedge(x\wedge z)=(x\wedge y)\wedge z$;

(c) $(x\wedge y)\wedge((x\wedge z)\wedge(x\wedge u))=((x\wedge y)\wedge(x\wedge z))\wedge(x\wedge u)$.

We refer the reader to the papers of Nambooripad [9], [10] and Meakin and Pastijn [6], [7] for a discussion of the equivalence between these methods of viewing local semilattices and for all relevant notation and terminology. In particular, we regard the ω^l and ω^r relations in a local semilattice $(E,\wedge)$ as being defined in such a way that the binary operation $\wedge$ extends the basic products. With this convention we have, for $e,f\in E$,

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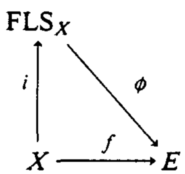
$$e\,\omega^r f \quad \text{if and only if} \quad e=f\wedge e,$$

$$e\,\omega^l f \quad \text{if and only if} \quad e=e\wedge f,$$

$\omega^l(e)\cap\omega^r(f)=\omega(f\wedge e)$ (where $\omega=\omega^l\cap\omega^r$), $(\omega^l(e),\wedge)$ is a left normal band, $(\omega^r(e),\wedge)$ is a right normal band and $(\omega(e),\wedge)$ is a semilattice. The binary operation $\wedge$ is not in general associative: in fact $\wedge$ is associative if and only if $(E,\wedge)$ is a normal band (Schein [13]). Consequently, it will be convenient to denote $e\wedge f$ either by $e\varrho_f$ or $f\lambda_e$ on occasions.

A structure theorem for local semilattices in terms of semilattices and sets was given in [6]. Definition (iv) above makes it clear that local semilattices form a variety and so free local semilattices exist. The free local semilattice on a set X is a pair (FLS_X, i) where FLS_X is a local semilattice and i is a mapping $i: X\to\mathrm{FLS}_X$ such that, for each local semilattice E and mapping $f: X\to E$, there is a unique homomorphism $\phi:\mathrm{FLS}_X\to E$ such that the diagram



commutes. The free local semilattice on two generators was described by Meakin and Pastijn in [7] and its images (i.e. all local semilattices on two generators) were described by Meakin in [4]. In the present paper we provide a construction of the free local semilattice on an arbitrary set. Our construction depends on Scheiblich's construction [12] of the semilattice of idempotents of the free inverse monoid on a set. We describe his construction briefly below.

Let X be a set and X' a set of the same cardinality as X with $X\cap X'=\emptyset$ and $x\to x'$ a bijection from X onto X'. Identify $(x')'$ with x for each $x\in X$. A word $w=x_1x_2,\dots,x_n$ of the free semigroup on $X\cup X'$ is called *reduced* if $x_i\neq x'_{i+1}$ for any i: we denote the set of all reduced words by R. The multiplication in the free group $R\cup\{1\}$ will be denoted by $\cdot$ if necessary, to distinguish it from the multiplication in the free monoid which will be denoted by concatenation. A finite subset T of $R\cup\{1\}$ is called *closed* if $1\in T$ and $w=x_1x_2\dots x_n\in T$ implies $x_1\dots x_i\in T$ for all $i\leq n$. The following result is due to Scheiblich [12].

Proposition 1.1. *Relative to the operation of taking unions of sets, the set of all finite closed subsets of $R\cup\{1\}$ forms a semilattice E_X isomorphic to the semilattice of idempotents of the free inverse monoid on X.*

We shall use the notation developed in this construction of E_X throughout the paper.

2. The construction

Let X be a non-empty set and $X_0 = X \cup \{x_0\}$ where $x_0 \notin X \cup X'$. As above, E_X denotes the semilattice of idempotents of the free inverse monoid on X. Denote the principal ideal of E_X generated by $A \in E_X$ by $\langle A \rangle$: thus $\langle A \rangle = \{B \in E_X : B \supseteq A\}$. Note that the antiatoms of E_X (elements covered by $\{1\}$) are of the form $\{1, x\}$ or $\{1, x'\}$ for $x \in X$. For $x, y \in X$ let $\pi(x, y)$ be the 2-cycle of S_X (the symmetric group on X) which interchanges x and y ($\pi(x, y)$ is the identity map on X if $x = y$) and let $\pi(x, y)$ act on a word $w = x_1 \dots x_n \in R \cup \{1\}$ by $w\pi(x, y) = x_1\pi(x, y) \dots x_n\pi(x, y)$; where $z\pi(x, y)$ denotes the action of $\pi(x, y)$ on z if $z \in X$ and $z'\pi(x, y) = (z\pi(x, y))'$ if $z \in X$. Note that, if $w \in R$, then $w\pi(x, y) \in R$. Extend the action of $\pi(x, y)$ to E_X by defining $A\pi(x, y) = \{w\pi(x, y) : w \in A\}$ for $A \in E_X$: note that $A\pi(x, y) \in E_X$ if $A \in E_X$. For $x \in X \cup X'$ define $\bar{x} : R \cup \{1\} \to R \cup \{1\}$ by $w\bar{x} = x \cdot w$ for $w \in R \cup \{1\}$ and define $A\bar{x} = \{w\bar{x} : w \in A\}$ for $A \in E_X$. Now define a principal ideal isomorphism $\gamma_{x,y} : \langle\{1, x\}\rangle \to \langle\{1, y'\}\rangle$ (for $x, y \in X$) by $\gamma_{x,y} = \pi(x, y)\bar{y}' = \bar{x}'\pi(x, y)$; i.e. $A\gamma_{x,y} = A\pi(x, y)\bar{y}' = A\bar{x}'\pi(x, y)$, for $A \in \langle\{1, x\}\rangle$. Note that $\gamma_{x,y}$ is a principal ideal isomorphism from $\langle\{1, x\}\rangle$ onto $\langle\{1, y'\}\rangle$ and so $\gamma_{xy} \in T_{E_X}$ (the Munn semigroup [8] of E_X): the inverse of $\gamma_{x,y}$ in T_{E_X} is the principal ideal isomorphism $\gamma_{x,y}^{-1} = \bar{y}\pi(x, y) = \pi(x, y)\bar{x}$ of $\langle\{1, y'\}\rangle$ onto $\langle\{1, x\}\rangle$. Finally let ι denote the identity automorphism of E_X and for $x, y \in X_0$ we let

$$p_{xy} = \begin{cases} \iota & \text{if } x = y, \\ \gamma_{xx} & \text{if } y = x_0 \text{ and } x \neq x_0, \\ \gamma_{yy} & \text{if } x = x_0 \text{ and } y \neq x_0, \\ \gamma_{yx} & \text{if } x \neq y \text{ and } x, y \in X. \end{cases} \tag{1}$$

We are now in a position to state the main theorem of the paper.

Theorem 2.1. *Let X be a non-empty set, $x_0 \notin X \cup X'$ and let $P = (p_{x,y})$ be the $X_0 \times X_0$ matrix with (x, y) entry $p_{xy} \in T_{E_X}$ defined by* (1). *Form the $X_0 \times X_0$ Rees matrix semigroup $\mathscr{M} = \mathscr{M}(T_{E_X}; X_0, X_0; P)$ over T_{E_X}. Then $\mathscr{M}$ is a locally inverse semigroup whose biordered sets $\mathrm{FLS}_{X_0} = E(\mathscr{M})$ is a local semilattice. Define a map $i : X_0 \to \mathrm{FLS}_{X_0}$ by $i(x) = (x, \iota, x)$ for $x \in X_0$. Then the pair $(\mathrm{FLS}_{X_0}, \iota)$ is a free local semilattice on X_0.*

We prove this theorem by a sequence of lemmata.

Lemma 2.2. *The Rees matrix semigroup $\mathscr{M}$ is a locally inverse semigroup (and hence $E(\mathscr{M})$ is a local semilattice).*

Proof. Since T_{E_X} is a regular (in fact inverse) semigroup with identity ι and since P has an entry ι in each row and each column, it follows immediately from Márki [2] that $\mathscr{M}$ is a regular semigroup. Now if S is any inverse semigroup and P any $\Lambda \times I$ matrix over S, it is routine to check that, for idempotents (i, x, λ) and (j, y, μ) of $\mathscr{M}(S; I, \Lambda; P)$, we have $(i, x, \lambda)\,\omega\,(j, y, \mu)$ iff $i = j$, $\lambda = \mu$ and $x \leq y$ in S, $(i, x, \lambda)\,\mathscr{R}$

(j, y, μ) iff $i=j$ and $x \mathscr{R} y$ in S, and $(i, x, \lambda) \mathscr{L} (j, y, \mu)$ iff $\lambda=\mu$ and $x \mathscr{L} y$ in S. From this it follows easily that the regular part of $\mathscr{M}(S; I, \Lambda; P)$ is locally inverse. (This argument follows from results in McAlister [3] and is explicitly stated in Meakin [5].) Hence $\mathscr{M}$ is locally inverse.

A convenient description of the idempotents of $\mathscr{M}$ is provided in the next result.

Lemma 2.3. *Let $x, y \in X$ with $x \neq y$ and let $\gamma \in T_{E_X}$. Then we have:*
$(x, \gamma, x) \in E(\mathscr{M})$ if and only if $\gamma = \iota_{\langle A \rangle}$ for some $A \in E_X$;
$(x_0, \gamma, y) \in E(\mathscr{M})$ iff $\gamma = \bar{y}|_{\langle A \rangle}$ for some $A \in E_X$ with $y' \in A$;
$(x, \gamma, x_0) \in E(\mathscr{M})$ iff $\gamma = \bar{x}|_{\langle A \rangle}$ for some $A \in E_X$ with $x' \in A$; and
$(x, \gamma, y) \in E(\mathscr{M})$ iff $\gamma = (\bar{y}\pi(x, y))|_{\langle A \rangle}$ for some $A \in E_X$ with $y' \in A$.

Proof. For any $x, y \in X_0$ and $\gamma \in T_{E_X}$ it is easy to see that $(x, \gamma, y) \in E(\mathscr{M})$ if and only if $\gamma \leq p_{yx}^{-1}$; i.e. $\gamma = p_{yx}^{-1}|_{\langle A \rangle}$ where A is some finite closed subset of R^1 such that $\langle A \rangle$ is contained in the domain of p_{yx}^{-1}. The result now follows easily from the definition (1) of the p_{xy}.

The above result shows that an element $(x, \gamma, y) \in \mathscr{M}$ is an idempotent if and only if γ is the restriction of p_{yx}^{-1} to a suitable principal ideal $\langle A \rangle$ of E_X: *it is convenient to denote such an idempotent by $(x, \gamma(A), y)$ in the remainder of this paper.* The Green's $\mathscr{R}$ and $\mathscr{L}$ relations on $E(\mathscr{M})$ are described in terms of this notation in the following lemma.

Lemma 2.4. *If $x, y \in X_0$ and $x \neq y$, then $(x, \gamma(A), x) \mathscr{R} (x, \gamma(B), y)$ iff $A = B$; if $x, y \in X$ and $x \neq y$ then $(y, \gamma(A), y) \mathscr{L} (x, \gamma(B), y)$ iff $A = B\bar{y}\pi(x, y)$; if $x, y \in X$ then:*
$(x_0, \gamma(A), y) \mathscr{L} (y, \gamma(B), y)$ iff $A\bar{y} = B$; and
$(x_0, \gamma(A), x_0) \mathscr{L} (x, \gamma(B), x_0)$ iff $A = B\bar{x}$.

Proof. Suppose that $(x, \gamma(A), x) \mathscr{R} (x, \gamma(B), y)$ with $x \neq y$ and $x, y \in X$. Then

$$(x, \iota_{\langle A \rangle}, x)(x, (\bar{y}\pi(x, y))|_{\langle B \rangle}, y) = (x, (\bar{y}\pi(x, y))|_{\langle B \rangle}, y),$$

so $\iota_{\langle A \rangle}(\bar{y}\pi(x, y))|_{\langle B \rangle} = (\bar{y}\pi(x, y))|_{\langle B \rangle}$. Since the domain of the mapping on the left of this equation is $\langle A \cup B \rangle$ and the domain of the mapping on the right is B, we have $A \cup B = B$; i.e. $A \subseteq B$. Also,

$$(x, (\bar{y}\pi(x, y))|_{\langle B \rangle}, y)(x, \iota_{\langle A \rangle}, x) = (x, \iota_{\langle A \rangle}, x),$$

so $\bar{y}\pi(x, y)|_{\langle B \rangle}\pi(x, y)\bar{y}'\iota_{\langle A \rangle} = \iota_{\langle A \rangle}$ and hence $\iota_{\langle B \rangle}\iota_{\langle A \rangle} = \iota_{\langle A \rangle}$; i.e. $A \cup B = A$ and so $B \subseteq A$. It follows that $A = B$. One easily checks that conversely, if $A = B$ then $(x, \gamma(A), x) \mathscr{R} (x, \gamma(B), y)$. A similar argument applies if either $x = x_0$ or $y = x_0$.

Suppose now that $(y, \gamma(A), y) \mathscr{L} (x, \gamma(B), y)$ with $x \neq y$ and $x, y \in X$. Then

$$(y, \iota_{\langle A \rangle}, y)(x, (\bar{y}\pi(x, y))|_{\langle B \rangle}, y) = (y, \iota_{\langle A \rangle}, y),$$

so $\iota_{\langle A\rangle}\pi(x, y)\bar{y}'(\bar{y}\pi(x, y))\,|_{\langle B\rangle} = \iota_{\langle A\rangle}$ and it follows (by equating domains of these mappings) that $A \cup B\bar{y}\pi(x, y) = A$; i.e. $B\bar{y}\pi(x, y) \subseteq A$. Also,

$$(x, (\bar{y}\pi(x, y))\,|_{\langle B\rangle}, y)(y, \iota_{\langle A\rangle}, y) = (x, (\bar{y}\pi(x, y))\,|_{\langle B\rangle}, y),$$

so $(\bar{y}\pi(x, y))\,|_{\langle B\rangle}\iota_{\langle A\rangle} = (\bar{y}\pi(x, y))\,|_{\langle B\rangle}$ and so (by equating ranges of the mappings) we have $B\bar{y}\pi(x, y) \cup A = B\bar{y}\pi(x, y)$; i.e. $A \subseteq B\bar{y}\pi(x, y)$. Hence $A = B\bar{y}\pi(x, y)$. Again, one routinely verifies the converse: $(y, \gamma(A), y)\,\mathscr{L}\,(x, \gamma(B), y)$ if $A = B\bar{y}\pi(x, y)$. A similar argument applies when either $x = x_0$ or $y = x_0$.

For each $x \in X_0$ let b_x be the idempotent $b_x = (x, \iota, x)$ of $E(\mathscr{M})$ and let $b_0 = b_{x_0}$. The following result follows from Lemma 2.4.

Corollary 2.5. *For $x, y \in X$ with $x \neq y$, we have*

$$(x, \gamma(A), x)\varrho_{b_y}\lambda_{b_y} = (y, \gamma(\{1\} \cup A\bar{y}\pi(x, y)), y),$$

$$(x, \gamma(A), x)\lambda_{b_y}\varrho_{b_y} = (y, \gamma(\{1\} \cup A\bar{y}'\pi(x, y)), y),$$

$$(x, \gamma(A), x)\varrho_{b_0}\lambda_{b_0} = (x_0, \gamma(\{1\} \cup A\bar{x}), x_0),$$

$$(x, \gamma(A), x)\lambda_{b_0}\varrho_{b_0} = (x_0, \gamma(\{1\} \cup A\bar{x}'), x_0),$$

$$(x_0, \gamma(A), x_0)\varrho_{b_y}\lambda_{b_y} = (y, \gamma(\{1\} \cup A\bar{y}), y),$$

$$(x_0, \gamma(A), x_0)\lambda_{b_y}\varrho_{b_y} = (y, \gamma(\{1\} \cup A\bar{y}'), y).$$

Proof. By Lemma 2.3, if $x \neq y$, we have $(x, \gamma(A), x)\varrho_{b_y} = (x, \gamma(A), x) \wedge (y, \iota, y) = (x, \gamma(C), y)$, where C is the smallest finite closed subset of $R \cup \{1\}$ which contains A and y'. Hence $(x, \gamma(A), x)\varrho_{b_y} = (x, \gamma(\{y'\} \cup A), y)$. It follows from Lemma 2.4 that

$$(x, \gamma(A), x)\varrho_{b_y}\lambda_{b_y} = (y, \gamma((\{y'\} \cup A)\bar{y}\pi(x, y)), y) = (y, \gamma(\{1\} \cup A\bar{y}\pi(x, y)), y),$$

as required. The other results in the statement of Corollary 2.5 follow in a similar fashion.

We aim to show that $E(\mathscr{M})$ is generated (as a local semilattice) by $\{b_x : x \in X_0\}$. We introduce the following notation in order to do this. For $x \in X_0$ define maps $\alpha(x)$ and $\alpha(x') : E(\mathscr{M}) \to E(\mathscr{M})$ by

$$\alpha(x) = \varrho_{b_x}\lambda_{b_x} \quad \text{and} \quad \alpha(x') = \lambda_{b_x}\varrho_{b_x}. \tag{2}$$

[Here x_0' is an element not in $X_0 \cup X'$: we denote $X' \cup \{x_0'\}$ by X_0'.] For $x \in X_0$ and $y \in X$ define

$$[x, y] = \begin{cases} x_0 & \text{if } x = y, \\ y & \text{if } x \neq y. \end{cases} \tag{3}$$

If $y_1, y_2, \ldots, y_p \in X \cup X'$ we let $I(y_1 y_2 \ldots y_p)$ denote the set of initial segments of

$y_1y_2\dots y_p$; i.e.

$$I(y_1y_2\dots y_p)=\{1, y_1, y_1y_2, \dots, y_1y_2\dots y_p\}.$$

Clearly $I(y_1y_2\dots y_p)$ is a finite closed subset of $R\cup\{1\}$; i.e. $I(y_1y_2\dots y_p)\in E_X$.

Lemma 2.6. *If $x\in X_0$ and $y_1, y_2, \dots, y_p\in X\cup X'$, then there are elements $z_1, z_2, \dots, z_p\in X_0\cup X_0'$ and $z_0\in X_0$ such that $z_i\notin\{z_{i+1}, z'_{i+1}\}$ for $i=0,1,\dots,p-1$ and $(x,\gamma(I(y_1y_2\dots y_p)),x)=b_{z_0}\alpha(z_1)\alpha(z_2)\dots\alpha(z_p)$.*

Proof. We do this by induction on p. If $p=1$, then $(x,\gamma(I(y_1y_2\dots y_p)),x)=(x,\gamma(\{1,y_1\}),x)$. If $y_1\in X$ set $z_0=[x,y_1]$ and $z_1=x$: if $y_1\in X'$ set $z_0=[x,y_1']$ and $z_1=x'$. One easily checks from Corollary 2.5 that $(x,\gamma(\{1,y_1\}),x)=b_{z_0}\alpha(z_1)$ in all cases: for example, $(x_0,\gamma(\{1,y\}),x_0)=(y,\iota,y)\alpha(x_0)$ if $y\in X$, and all other cases may be checked similarly. It is also clear that $z_0\notin\{z_1,z_1'\}$. Suppose now that the result is true for all words $y_1y_2\dots y_k$ of length $k<p$ and consider $(x,\gamma(I(y_1y_2\dots y_p)),x)$ with $x\neq x_0$. Define $w\in E(\mathcal{M})$ by

$$w=\begin{cases}([x,y_1],\gamma(I(y_2\dots y_p\pi(x,y_1))),[x,y_1]) & \text{if } y_1\in X,\\ ([x,y_1'],\gamma(I(y_2\dots y_p\pi(x,y_1'))),[x,y_1']) & \text{if } y_1\in X',\end{cases}$$

and let

$$z_p=\begin{cases}x & \text{if } y_1\in X,\\ x' & \text{if } y_1\in X'.\end{cases}$$

Clearly $z_p\notin\{[x,y_1],[x,y_1'],[x,y_1]',[x,y_1']'\}$. Now if $y_1=x$, then $[x,y_1]=x_0$ and $\pi(x,y_1)$ is the identity transposition, so $w=(x_0,\gamma(I(y_2\dots y_p),x_0)$ and so by Corollary 2.5,

$$\begin{aligned}w\alpha(z_p)=w\alpha(x)&=(x,\gamma(\{1\}\cup(I(y_2\dots y_p))\bar{y}_1),x)\\&=(x,\gamma(\{1\}\cup y_1\,.\,I(y_2\dots y_p)),x)\\&=(x,\gamma(I(y_1y_2\dots y_p)),x).\end{aligned}$$

A similar argument deals with the case $y_1=x'$. If $y_1\in X$ and $y_1\neq x$, then $[x,y]=y_1$, so $w=(y_1,\gamma(I(y_2\dots y_p\pi(x,y_1)),y_1)$ and so by Corollary 2.5,

$$\begin{aligned}w\alpha(z_p)=w\alpha(x)&=(x,\gamma(\{1\}\cup I(y_2\dots y_p\pi(x,y_1))\bar{x}\pi(x,y_1),x)\\&=(x,\gamma(\{1\}\cup(x\,.\,I(y_2\dots y_p\pi(x,y_1)))\pi(x,y_1)),x)\\&=(x,\gamma(\{1\}\cup I(y_1y_2\dots y_p)),x)\\&=(x,\gamma(I(y_1y_2\dots y_p)),x).\end{aligned}$$

The case $y_1\in X'$ and $y_1'\neq x$ is dealt with similarly. Hence, in all cases, if $x\neq x_0$ we have $(x,\gamma(I(y_1y_2\dots y_p)),x)=w\alpha(z_p)$. If $x=x_0$ define $w\in E(\mathcal{M})$ by

$$w=\begin{cases}(y_1,\gamma(I(y_2\dots y_p)),y_1) & \text{if } y_1\in X,\\ (y_1',\gamma(I(y_2\dots y_p)),y_1') & \text{if } y_1\in X',\end{cases}$$

and let $z_p = x_0$ if $y_1 \in X$ and $z_p = x_0'$ if $y_1 \in X'$. Then in both cases it is easy to check that $(x_0, \gamma(I(y_1 y_2 \dots y_p)), x_0) = w\alpha(z_p)$: we thus have $(x, \gamma(I(y_1 y_2 \dots y_p)), x) = w\alpha(z_p)$ in all cases. Now by the induction hypothesis there exist $z_1, z_2, \dots, z_{p-1} \in X_0 \cup X_0'$ and $z_0 \in X_0$ such that $z_i \notin \{z_{i+1}, z_{i+1}'\}$ for $i = 0, 1, \dots, p-2$ and $w = b_{z_0}\alpha(z_1)\alpha(z_2)\dots\alpha(z_{p-1})$. It follows that $(x, \gamma(I(y_1 y_2 \dots y_p)), x) = b_{z_0}\alpha(z_1)\alpha(z_2)\dots\alpha(z_{p-1})\alpha(z_p)$; since $z_{p-1} = [x, y_1]$ or $[x, y_1']$, $z_{p-1} \notin \{z_p, z_p'\}$ and hence the result follows by induction.

Lemma 2.7. *The representation in the previous lemma is unique; i.e. if*

$$b_{z_0}\alpha(z_1)\alpha(z_2)\dots\alpha(z_p) = b_{u_0}\alpha(u_1)\alpha(u_2)\dots\alpha(u_q)$$

for some $z_0, u_0 \in X_0$ *and* $z_i, u_j \in X_0 \cup X_0'$ *with* $z_i \notin \{z_{i+1}, z_{i+1}'\}$ *and* $u_i \notin \{u_{i+1}, u_{i+1}'\}$, *then* $p = q$ *and* $z_i = u_i$ *for* $i = 0, \dots, p$.

Proof. It is straightforward to verify from Corollary 2.5 that

$$b_{z_0}\alpha(z_1)\alpha(z_2)\dots\alpha(z_j) = (y_j, \gamma(I(w_j)), y_j), \quad \text{for } j = 1, \dots, p,$$

where the elements y_j of X_0 and words w_j in the free semigroup on $X \cup X'$ are given inductively as follows:

$$y_j = \begin{cases} z_j & \text{if } z_j \in X_0, \\ z_j' & \text{if } z_j \in X_0', \end{cases} \quad \text{for } j = 1, \dots, p, \tag{4}$$

$$w_1 = \begin{cases} z_1 & \text{if } z_0 = x_0, \\ z_0 & \text{if } z_0 \neq x_0 \text{ and } z_1 \in X_0, \\ z_0' & \text{if } z_0 \neq x_0 \text{ and } z_1 \in X_0', \end{cases} \tag{5}$$

and for $j = 1, \dots, p-1$,

$$w_{j+1} = \begin{cases} z_{j+1} w_j & \text{if } y_j = x_0, \\ (z_{j+1} w_j)\pi(z_{j+1}, y_j) & \text{if } y_j \neq x_0 \text{ and } z_{j+1} \in X, \\ (z_{j+1} w_j)\pi(z_{j+1}', y_j) & \text{if } y_j \neq x_0 \text{ and } z_{j+1} \in X', \\ y_j w_j & \text{if } y_j \neq 0 \text{ and } z_{j+1} = x_0, \\ y_j' w_j & \text{if } y_j \neq 0 \text{ and } z_{j+1} = x_0'. \end{cases} \tag{6}$$

Similarly one sees that

$$b_{u_0}\alpha(u_1)\alpha(u_2)\dots\alpha(u_j) = (a_j, \gamma(I(v_j)), a_j), \quad \text{for } j = 1, \dots, q,$$

where the elements a_j of X_0 and the words v_j ($j = 1, \dots, q$) are obtained from $u_0, u_1, \dots, u_q$ in the same manner as the y_j and w_j ($j = 1, \dots, p$) are obtained from $z_0, z_1, \dots, z_p$. It follows immediately that $w_p = v_q$ and hence that $p = q$ since p is the length of w_p and q is the length of v_q: it is thus also immediate that $y_p = a_p$. We prove the desired result by induction on p. Suppose first that $p = 1$. If $u_0 = x_0$ and $u_1 \in X$, then we see that $w_1 = v_1 = u_1 \in X$ and $y_1 = a_1 = u_1$: since $w_1 = y_1$ we see from (4) and (5) that we must have $z_0 = x_0$ and so $z_0 = u_0$: also since $z_0 = x_0$ we must have $z_1 = w_1$ so $z_1 = u_1$. If $u_0 = x_0$ and $u_1 \in X'$, then we see that $w_1 = v_1 = u_1 \in X'$ and

$y_1 = a_1 = u_1'$: since $y_1 = w_1'$ we see from (4) and (5) that $z_0 = x_0$ and so $z_0 = u_0$: also since $z_0 = x_0$ we have $z_1 = w_1$ and so $z_1 = u_1$. If $u_0 \in X$ and $u_1 = x_0$, then $w_1 = v_1 = u_0 \in X$ and $y_1 = a_1 = u_1 = x_0$: since $w_1 \notin \{y_1, y_1'\}$ and $w_1 \in X$ we see from (4) and (5) that $w_1 = z_0$ and so $z_0 = u_0$: also $z_1 \in X_0$, so $y_1 = z_1$ and so $z_1 = u_1$. If $u_0 \in X$ and $u_1 = x_0'$, then $w_1 = v_1 = u_0' \in X'$ and $y_1 = a_1 = u_1' = x_0$: since $w_1 \notin \{y_1, y_1'\}$ and $w_1 \in X'$ we see from (4) and (5) that $w_1 = z_0'$ and so $z_0 = u_0$: also $z_1 \in X_0'$, so $y_1 = z_1'$ and so $z_1 = u_1$. If $u_0 \in X$ and $u_1 \in X$, then $w_1 = v_1 = u_0 \in X$ and $y_1 = a_1 = u_1 \in X$: since $w_1 \notin \{y_1, y_1'\}$ and $w_1 \in X$ we see from (4) and (5) that $w_1 = z_0$ and so $z_0 = u_0$: also $z_1 \in X_0$ so $y_1 = z_1$ and so $z_1 = a_1$. The remaining case ($u_0 \in X$ and $u_1 \in X'$) is dealt with in a similar fashion and so $z_0 = u_0$ and $z_1 = u_1$ in all cases: thus the result is true for $p = 1$.

Suppose inductively that the result is true for $p = j$ and assume that

$$b_{z_0}\alpha(z_1)\ldots\alpha(z_{j+1}) = b_{u_0}\alpha(u_1)\ldots\alpha(u_{j+1})$$

for z_i, u_i as in the statement of the lemma. In the notation of the beginning of the proof of the lemma, we have $w_{j+1} = v_{j+1}$ and $y_{j+1} = a_{j+1}$. If $u_j = x_0$ and $u_{j+1} \in X$, then $a_j = u_j = x_0$ and so $w_{j+1} = v_{j+1} = u_{j+1}v_j$ and $y_{j+1} = a_{j+1} = u_{j+1}$: from (4) and (6) it follows that $y_j = x_0$, so $y_j = a_j$ and $w_{j+1} = z_{j+1}w_j$: hence (since $w_{j+1} = v_{j+1}$), $z_{j+1} = u_{j+1}$ and $v_j = w_j$. If $u_j = x_0$ and $u_{j+1} \in X'$, then $a_j = u_j = x_0$ and so $w_{j+1} = v_{j+1} = u_{j+1}v_j$ and $y_{j+1} = a_{j+1} = u_{j+1}'$: from (4) and (6) it follows that $y_j = x_0$, so $y_j = a_j$ and $w_{j+1} = z_{j+1}w_j$: hence, again it follows that $z_{j+1} = u_{j+1}$ and $v_j = w_j$. If $u_j \in X$ and $u_{j+1} = x_0$, then $a_j = u_j \neq x_0$ and so $w_{j+1} = v_{j+1} = u_jv_j$ and $y_{j+1} = a_{j+1} = u_{j+1}$: since the first letter in w_{j+1} is $u_j \notin \{y_{j+1}, y_{j+1}'\}$ and $u_j \in X$, we see from (4) and (6) that we must have $y_j \neq x_0$ and $z_{j+1} \in X_0$, so $z_{j+1} = y_{j+1} = a_{j+1} = u_{j+1} = x_0$ and so $y_jw_j = w_{j+1} = v_{j+1} = u_jv_j$, from which it follows that $y_j = u_j = a_j$ and $w_j = v_j$. If $u_j \in X$ and $u_{j+1} = x_0$ then a similar argument shows again that $z_{j+1} = u_{j+1}$, $w_j = v_j$ and $y_j = a_j$. If $u_j \in X$ and $u_{j+1} \in X$ then $a_j = u_j \neq x_0$ and so $w_{j+1} = v_{j+1} = (u_{j+1}v_j)\pi(u_{j+1}, a_j)$ and $y_{j+1} = a_{j+1} = u_{j+1}$: since the first letter in w_{j+1} is $a_j = u_j$ and $u_j \notin \{y_{j+1}, y_{j+1}'\}$ and $u_j \in X$, we see from (4) and (6) that we must have $y_j \neq x_0$ and $z_{j+1} \in X_0$, so $z_{j+1} = y_{j+1} = a_{j+1} = u_{j+1} \in X$ and so $(z_{j+1}w_j)\pi(z_{j+1}, y_j) = w_{j+1} = v_{j+1} = (u_{j+1}v_j)\pi(u_{j+1}, a_j)$, from which it follows that $y_j = a_j$ and hence $w_j = v_j$. Finally, if $u_j \in X$ and $u_{j+1} \in X'$ then a similar argument shows that $z_{j+1} = u_{j+1}$, $y_j = a_j$ and $w_j = v_j$. Thus in all cases, we have $z_{j+1} = u_{j+1}$, $y_j = a_j$ and $w_j = v_j$. Since $y_j = a_j$ and $w_j = v_j$, we have

$$b_{z_0}\alpha(z_1)\alpha(z_2)\ldots\alpha(z_j) = b_{u_0}\alpha(u_1)\alpha(u_2)\ldots\alpha(u_j),$$

and hence by the induction assumption, $z_i = u_i$ for $i = 0, 1, \ldots, j$. We also have $z_{j+1} = u_{j+1}$ and so the result is proved by induction.

Lemma 2.8. *Every element $(x, \gamma(A), y)$ of $E(\mathscr{M})$ may be expressed in the form*

$$(x, \gamma(A), y) = \left[\bigwedge_{w \in A} (x, \gamma(I(w)), x)\right]\varrho_{b_y},$$

where

$$\bigwedge_{w \in A} (x, \gamma(I(w)), x)$$

denotes the meet in the semilattice $\omega(b_x)$ of the elements $(x, \gamma(I(w)), x)$, $w \in A$.

Proof. By Lemma 2.4 we see that $(x, \gamma(A), x) \mathscr{R} (x, \gamma(A), y)$ and so $(x, \gamma(A), y) = (x, \gamma(A), x)\varrho_{b_y}$. But by Lemma 2.3,

$$(x, \gamma(A), x) = (x, \iota_A, x) = \bigwedge_{w \in A} (x, \iota_{I(w)}, x) = \bigwedge_{w \in A} (x, \gamma(I(w)), x).$$

Corollary 2.9. *$E(\mathscr{M})$ is generated by $\{b_x : x \in X_0\}$.*

Proof. This is immediate from Lemma 2.8 and Lemma 2.6.

Remark 2.10. Lemmas 2.3, 2.6, 2.7 and 2.8 provide us with a 'canonical form' for expressing elements of $E(\mathscr{M})$ in terms of the generators b_x, $x \in X_0$. Every element of $E(\mathscr{M})$ is uniquely expressible in the form $(x, \gamma(A), y)$ for some $x, y \in X_0$ and $A \in E_x$ and hence may be uniquely expressed in the form $[\bigwedge_{w \in A}(x, \gamma(I(w)), x)]\varrho_{b_y}$ (where we interpret $\gamma(I(w))$ as ι if $A = \{1\}$) – and every element $(x, \gamma(I(w)), x)$ may be uniquely expressed in the form $b_{z_0}\alpha(z_1)\alpha(z_2) \dots \alpha(z_p)$ for some $z_0 \in X_0$ and $z_i \in X_0 \cup X_0'$. Of course this canonical form does not provide the only way of expressing $(x, \gamma(A), y)$ in terms of the generators b_x, $x \in X_0$: nor does it necessarily provide the 'shortest' way to generate $(x, \gamma(A), y)$ from $\{b_x : x \in X_0\}$.

In order to prove Theorem 2.1, we need to establish some notation and results which apply to an arbitrary local semilattice E. Let E be a local semilattice and let E' be a set disjoint from E such that the mapping $e \to e'$ is a bijection from E onto E': again, identify $(e')'$ with e for each $e \in E$. For $e \in E$ define $\alpha(e)$ and $\alpha(e') : E \to E$ by:

$$\alpha(e) = \varrho_e \lambda_e \quad \text{and} \quad \alpha(e') = \lambda_e \varrho_e. \tag{7}$$

Lemma 2.11. *If E is any local semilattice and $e, f \in E$, the mappings $\alpha(e)|_{\omega(f)}$ and $\alpha(e')|_{\omega(f)}$ are homomorphisms from $\omega(f)$ into $\omega(e)$; also $f\alpha(e') \mathscr{R} e\varrho_f \mathscr{L} e\alpha(f)$ and if $t \in \omega(f\alpha(e'))$, $s \in \omega(e\alpha(f))$, then $t \mathscr{R} t\varrho_f \mathscr{L} t\alpha(f)$ and $s \mathscr{L} s\lambda_e \mathscr{R} s\alpha(e')$. In addition, if $g \in \omega(e)$ and $h \in \omega(f)$ then $g\varrho_h = [g \wedge (h\alpha(e'))]\varrho_f$.*

Proof. Since $e\varrho_f = f\lambda_e = e \wedge f \in \omega^l(f) \cap \omega^r(e)$ it is clear that there is some element in $R_{e \wedge f} \cap \omega(e)$ and so $e\varrho_f = f\lambda_e \mathscr{R} f\lambda_e\varrho_e = f\alpha(e')$: similarly, $e\varrho_f \mathscr{L} e\alpha(f)$. If $t \in \omega(f\alpha(e'))$, then there is an element in $R_t \cap \omega(e\varrho_f)$ and so $t \mathscr{R} t\varrho_f$: again, $t\varrho_f \,\omega\, e\varrho_f$ and $e\varrho_f \mathscr{L} e\alpha(f)$, so $t\varrho_f \mathscr{L} t\alpha(f)$. Similarly, if $s \in \omega(e\alpha(f))$, then $s \mathscr{L} s\lambda_e \mathscr{R} s\alpha(e')$. Now let $h_1, h_2 \in \omega(f)$ and $h = h_1 \wedge h_2$ (the meet in $\omega(f)$). Since $h \,\omega\, h_i$ it is clear from the definition of the $\wedge$ operation in a local semilattice that $h\lambda_e \,\omega\, h_i\lambda_e$ and $h\alpha(e') = h\lambda_e\varrho_e \,\omega\, h_i\lambda_e\varrho_e = h_i\alpha(e')$ for $i = 1, 2$. Now suppose that $k \,\omega\, h_i\alpha(e')$ for $i = 1, 2$. Then by the same argument as above $k\alpha(f) \,\omega\, h_i\alpha(e')\alpha(f) \,\omega\, h_i$ for $i = 1, 2$, so $k\alpha(f) \,\omega\, h_1 \wedge h_2 = h$ and it follows that $k\alpha(f)\alpha(e') \,\omega\, h\alpha(e')$. But for $i = 1, 2$, $k \,\omega\, h_i\alpha(e') \mathscr{R} h_i\lambda_e$, so $k\varrho_f \mathscr{R} k$. Also $k\varrho_f \,\omega\, h_i\lambda_e \mathscr{L} h_i\lambda_e\lambda_f$, so $k\varrho_f\lambda_f \mathscr{L} k\varrho_f$. Hence $k \mathscr{R} k\varrho_f \mathscr{L} k\varrho_f\lambda_f = k\alpha(f)$. It follows that $k\alpha(f)\alpha(e') = k$ and hence $k \,\omega\, h\alpha(e')$. Hence $h\alpha(e') = h_1\alpha(e') \wedge$

$h_2\alpha(e')$, and so $\alpha(e')$ is a homomorphism as required. Similarly $\alpha(e)$ is a homomorphism.

Now let $g\in\omega(e)$ and $h\in\omega(f)$. Then

$$
\begin{aligned}
g\varrho_h = g\wedge h &= (e\wedge g)\wedge h \\
&= (e\wedge g)\wedge(e\wedge h) \quad \text{(by axiom (b) for local semilattices)} \\
&= g\wedge e\varrho_h \,\mathscr{R}\, g\wedge h(\alpha(e')) \quad (\text{since } e_{\varrho_h} \mathscr{R}\, h\alpha(e')),
\end{aligned}
$$

so $g\varrho_h \,\mathscr{R}\, g\wedge(h\alpha(e'))$ and it follows that $g\varrho_h = [g\wedge(h\alpha(e'))]\varrho_f$, as required.

We are now in a position to provide the following.

Proof of Theorem 2.1. Let $f: X_0\to E$ be any map from X_0 to the local semilattice E. We must show that there is a unique homomorphism $\phi: E(\mathscr{M})\to E$ such that $\phi(x,\iota,x)=f(x)$ for each $x\in X_0$. In order to define the map $\phi: E(\mathscr{M})\to E$ we use the canonical form for elements of $E(\mathscr{M})$ (see Remark 2.10). If $(x,\gamma(A),y)\in E(\mathscr{M})$ then we write $(x,\gamma(A),y)=[\bigwedge_{w\in A}(x,\gamma(I(w)),x)]\varrho_{b_y}$: for each $w\in A$ there are elements $z_0\in X_0$ and $z_i\in X_0\cup X_0'$ $(i=1,\dots,p)$ such that $(x,\gamma(I(w)),x)=b_{z_0}\alpha(z_1)\dots\alpha(z_p)$. In this case we write $\phi((x,\gamma(I(w)),x))=f(z_0)\alpha(f(z_1))\dots\alpha(f(z_p))$, where $f(z_i)=[f(z_i')]'$ for $z_i\in X_0'$. Now since $b_{z_0}\alpha(z_1)\dots\alpha(z_p)\in\omega(b_x)$, we have

$$
z_p=\begin{cases} x & \text{if } z_p\in X_0,\\ x' & \text{if } z_p\in X_0',\end{cases}
$$

and so

$$
f(z_p)=\begin{cases} f(x) & \text{if } z_p\in X_0,\\ [f(x)]' & \text{if } z_p\in X_0',\end{cases}
$$

from which it follows that $f(z_0)\alpha(f(z_1))\dots\alpha(f(z_p))\in\omega(f(x))$. Hence $\phi((x,\gamma(I(w)),x))\in\omega(f(x))$ for each $w\in A$. Now define

$$
\phi((x,\gamma(A),y))=\left[\bigwedge_{w\in A}\phi((x,\gamma(I(w)),x))\right]\varrho_{f(y)}, \tag{8}
$$

where the meet operation is the meet operation in $\omega(f(x))$. Since each element of $E(\mathscr{M})$ has a unique canonical form, ϕ is a well-defined map from $E(\mathscr{M})$ to E: also $\phi(b_x)=\phi((x,\iota,x))=[f(x)]\varrho_{f(x)}=f(x)$ for each $x\in X_0$. We need to check that ϕ is a homomorphism.

Suppose first that $(x,\gamma(A),x)\,\mathscr{R}\,(x,\gamma(B),y)$ in $E(\mathscr{M})$. Then by Lemma 2.4 we have $A=B$. Suppose first that $x\neq y$ and $x,y\in X$, so that $y'\in A$ by Lemma 2.3. Now

$$
\begin{aligned}
\phi((x,\gamma(A),x)) &= \left[\bigwedge_{w\in A}\phi((x,\gamma(I(w)),x))\right]\varrho_{f(x)} \\
&= \bigwedge_{w\in A}\phi((x,\gamma(I(w)),x)),
\end{aligned}
$$

and $\phi((x,\gamma(A),y))=[\bigwedge_{w\in A}\phi((x,\gamma(I(w)),x))]\varrho_{f(y)}$. Since $y'\in A$ and since

$(x, \{1, y'\}, x) = b_y\alpha(x')$ (by Corollary 2.5), we see that $\phi((x, \gamma(A), x))\, \omega f(y)\alpha([f(x)]')$ in E (by (8)). But $f(y)\alpha([f(x)]')\, \mathscr{R} f(y)\lambda_{f(x)}$ so $\phi((x, \gamma(A), x))\, \mathscr{R}\, [\phi((x, \gamma(A), x))]\varrho_{f(y)} = \phi((x, \gamma(A), y))$ by Lemma 2.11. A similar argument in case $x \neq y$ and $x = x_0$ or $x \neq y$ and $y = x_0$ shows that $\phi((x, \gamma(A), x)\, \mathscr{R}\, \phi((x, \gamma(A), y))$ in these cases also. Hence we see that, if $(x, \gamma(A), x)\, \mathscr{R}\, (x, \gamma(B), y)$ in $E(\mathscr{M})$, then $\phi((x, \gamma(A), x))\, \mathscr{R}\, \phi((x, \gamma(B), y))$ in E.

Now suppose that $(x, \gamma(A), y)\, \mathscr{L} (y, \gamma(B), y)$ in $E(\mathscr{M})$. If $x \neq y$ and $x, y \in X$ then by Lemmas 2.3 and 2.4 we see that $y' \in A$ and $B = A\bar{y}\pi(x, y)$. Now

$$\phi((y, \gamma(B), y) = \left[\bigwedge_{w \in B} \phi((y, \gamma(I(w)), y))\right]\varrho_{f(y)}$$

$$= \left[\bigwedge_{w \in B} \phi((y, \gamma(I(w)), y))\right],$$

and

$$\phi((x, \gamma(A), y)) = \left[\bigwedge_{w \in A} \phi((x, \gamma(I(w)), x))\right]\varrho_{f(y)}.$$

As before, we have $y' \in A$ and $(x, \{1, y'\}, x) = b_y[\alpha(x)]'$, so

$$\bigwedge_{w \in A} \phi((x, \gamma(I(w)), x))\, \omega f(y)\alpha([f(x)]'),$$

i.e. $\phi((x, \gamma(A), x))\, \omega f(y)\alpha([f(x)]')$ in E. It follows by Lemma 2.11 that

$$\phi((x, \gamma(A), y)) = \phi((x, \gamma(A), x))\varrho_{f(y)}\, \mathscr{L}\, [\phi((x, \gamma(A), x))]\alpha(f(y)),$$

since $f(y)\alpha([f(x)]')\, \mathscr{R} f(y)\lambda_{f(x)}\, \mathscr{L} f(y)\lambda_{f(x)}\lambda_{f(y)}$. Hence

$$\phi((x, \gamma(A), y))\, \mathscr{L} \left[\bigwedge_{w \in A} \phi((x, \gamma(I(w)), x))\right]\alpha(f(y)).$$

But if $w \in A$ and $(x, \gamma(I(w)), x) = b_{z_0}\alpha(z_1) \ldots \alpha(z_p) \in \omega(x)$, then $(x, \gamma(I(w)), x)\alpha(y) = b_{z_0}\alpha(z_1) \ldots \alpha(z_p)\alpha(y) \in \omega(y)$ and $b_{z_0}\alpha(z_1) \ldots \alpha(z_p)\alpha(y) = (y, \gamma(I(v)), y)$ for some $v \in B$. It follows that

$$\left[\bigwedge_{w \in A} \phi((x, \gamma(I(w)), x))\right]\alpha(f(y) = \bigwedge_{w \in B} \phi((y, \gamma(I(v)), y))$$

$$= \phi((y, \gamma(B), y)),$$

and hence $\phi((x, \gamma(A), y))\, \mathscr{L} \phi((y, \gamma(B), y))$. A similar argument in the cases where $x = x_0$, $y \neq x$ and $y = x_0$, $y \neq x$ shows that $\phi((x, \gamma(A), y))\, \mathscr{L} \phi((y, \gamma(B), y))$ in E whenever $(x, \gamma(A), y)\, \mathscr{L} (y, \gamma(B), y)$ in $E(\mathscr{M})$. Hence ϕ preserves the $\mathscr{L}$- and $\mathscr{R}$-relations in $E(\mathscr{M})$.

Suppose now that $(x, \gamma(C), y), (u, \gamma(D), v) \in E(\mathscr{M})$. There are uniquely defined sets A and B such that $(x, \gamma(C), y) \wedge (u, \gamma(D), v) = (x, \gamma(A), x) \wedge (v, \gamma(B), v)$. If $x = v$, then it is evident from (8) that $\phi((x, \gamma(A), x) \wedge (v, \gamma(B), v)) = \phi((x, \gamma(A), x)) \wedge \phi((v, \gamma(B)), v))$, so assume $x \neq v$ and also that $x, v \in X$. Then by Lemmas 2.3 and 2.4, $(x, \gamma(A), x) \wedge (v, \gamma(B), v) = (x, \gamma(E), v)$, where $E = A \cup B\pi(x, v)\bar{v}' \cup \{v'\}$. We have

$$\phi((x,\gamma(A),x)) = \bigwedge_{w\in A} \phi((x,\gamma(I(w)),x)),$$

$$\phi((v,\gamma(B),v)) = \bigwedge_{t\in B} \phi((v,\gamma(I(t)),v)),$$

$$\phi((x,\gamma(E),v)) = \left[\bigwedge_{r\in E} \phi((x,\gamma(I(r)),x))\right]\varrho_{f(v)}.$$

By Lemma 2.11 we have $(x,\gamma(A),x)\wedge(v,\gamma(B),v) = [(x,\gamma(A),x)\wedge(v,\gamma(B),v)\alpha(x')]\varrho_{b_v}$, so $(x,\gamma(A),x)\wedge(v,\gamma(B),v)\alpha(x') = (x,\gamma(E),x)$, and hence

$$\left[\bigwedge_{w\in A} (x,\gamma(I(w)),x)\right]\wedge\left[\bigwedge_{t\in B} (v,\gamma(I(t)),v)\alpha(x')\right] = \bigwedge_{r\in E} (x,\gamma(I(r)),x).$$

It follows from (8) that

$$\phi((x,\gamma(A)),x))\wedge\phi((v,\gamma(B),v))\alpha([f(x)]') = \phi(x,\gamma(E),x),$$

and hence

$$\begin{aligned}\phi((x,\gamma(E),v)) &= \phi((x,\gamma(E),x))\varrho_{f(v)}\\ &= [\phi((x,\gamma(A),x))\wedge\phi((v,\gamma(B),v))\alpha([f(x)]')]\varrho_{f(v)}\\ &= \phi((x,\gamma(A),x))\wedge\phi((v,\gamma(B),v)) \quad \text{by Lemma 2.11.}\end{aligned}$$

Similar arguments in case $x=x_0$, $v\in X$ or $v=x_0$, $x\in X$ show that

$$\phi((x,\gamma(A),x)\wedge(v,\gamma(B),v)) = \phi((x,\gamma(A),x))\wedge\phi((v,\gamma(B),v))$$

in all cases. Since ϕ preserves $\mathscr{L}$- and $\mathscr{R}$-relations in $E(\mathscr{M})$, we have

$$\phi((x,\gamma(C),y)\wedge(u,\gamma(D),v)) = \phi((x,\gamma(C),y))\wedge\phi((u,\gamma(D),v))$$

for all $(x,\gamma(C),y),(u,\gamma(D),v)\in E(\mathscr{M})$. Hence ϕ is a homomorphism.

Finally, if ψ is any homomorphism from $E(\mathscr{M})$ to E such that $\psi(b_x)=f(x)$, then we must have $\psi(b_{z_0}\alpha(z_1)\dots\alpha(z_p)) = f(z_0)\alpha(f(z_1))\dots\alpha(f(z_p))$ for each $z_0\in X_0$ and $z_i\in X_0\cup X_0'$ $(i=1,\dots,p)$, so it easily follows that $\psi=\phi$. This establishes the uniqueness of ϕ and hence completes the proof of the theorem.

The explicit nature of the description of the free local semilattice provided in Theorem 2.1 makes it clear that the word problem is solvable in FLS_{X_0} for each set X_0. From Lemmas 2.3 and 2.4 one can explicitly compute the meets $(x,\gamma(A),y)\wedge(u,\gamma(B),v)$ in $E(\mathscr{M})$: hence any expression involving meets of the generators b_x of $E(\mathscr{M})$ may be reduced in a finite number of steps to the form $(x,\gamma(A),y)$ for some $x,y\in X_0$ and $A\in E_X$. It is thus possible to decide, in a finite number of steps, whether two expressions in the generators of $E(\mathscr{M})$ yield the same element of $E(\mathscr{M})$. Hence the word problem in FLS_{X_0} is solvable.

The local semilattice $E(\mathscr{M})$ is a disjoint union of its maximal subsemilattices in the sense of Pastijn [11] or Byleen, Meakin and Pastijn [1]. It is clear that the 'diagonal' maximal subsemilattices (those of the form $\{(x, \gamma(A), x) : A \in E_X\}$ for $x \in X_0$) are isomorphic to the semilattice E_X of idempotents of the free inverse monoid on X, while the other maximal subsemilattices are isomorphic to principal ideals generated by antiatoms of E_X. A diagram depicting the free local semilattice on two generators may be found in Meakin and Pastijn [7].

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